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Deformation quantization of bosonic strings

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Abstract. Deformation quantization of bosonic strings is considered. We show that the light-cone gauge is the most convenient classical description to perform the quantization of bosonic strings in the deformation quantization formalism. Similar to the field theory case, the oscillator variables greatly facilitates the analysis. The mass spectrum, propagators and the Virasoro algebra are finally described within this deformation quantization scheme.

1. Introduction

String theory is one of the most successful attempts to reconcile quantum mechanics and general relativity (for a review of the perturbative theory see [1–4]). From the physical point of view it is our best understanding of all matter and their interactions in an unified scheme. On the mathematical side, string theory has been used to motivate an unsuspected interplay among some mathematical subjects. At the perturbative level, it is well known that string theory is related to the theory of Riemann surfaces [5] and some aspects of algebraic geometry and mirror symmetry (see, for instance, [6]).

Non-perturbative revolution of string theory, through the introduction of D-branes and duality, has been shown to be related to some aspects of toric geometry (see, for instance, [7]), K-theory (see the Witten seminal paper, [8]), noncommutative geometry and deformation quantization theory [9–11]. This latter is relevant for the description of the low-energy effective theory of open strings on the D-brane world-volume, when a non-zero Neveu–Schwarz constant B field is introduced. Thus, in this context, deformation quantization describes properly the noncommutative spacetime instead of the standard quantization of the phase space of the two-dimensional conformal field theory.

Deformation quantization of phase space is an equivalent description to that of the operators and Hilbert spaces formalism. Although the operator theory is beautiful and has a rich structure, in the former one, the computations are easier because the relevant structure is still an algebra of functions with a well-defined star product (Moyal $*$ product).

The purpose of this paper is to provide explicitly the Weyl–Wigner–Moyal formalism to construct the Moyal $*$ product on the algebra of functions on the phase space of the bosonic string theory. This is the first step towards a complete description of the deformation quantization of the phase space for the interacting superstring theory and its further generalization to superstring field theory. Thus in this paper we focus on the application of the

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deformation quantization formalism to bosonic string theory. (The fermionic case is left for a forthcoming paper.) And at the same time this is the natural extension of the deformation quantization of classical fields developed in [12–18].

In order to be as self-contained as possible, in section 2, we briefly overview the preliminaries and notation of string theory in order to prepare the theory for quantization. Section 3 is devoted to the quantization of bosonic strings by using deformation quantization theory. The results of this section show that the deformation quantization of bosonic strings constitutes a new example of the application of the Weyl–Wigner–Moyal formalism. Indeed, it prepares the theory for further applications and the realization of more general formulations of deformation quantization (such as those given by Fedosov and Kontsevich [19]) to the phase space of string theory. In section 4 we describe the Casimir effect and the normal ordering of bosonic strings within deformation quantization formalism. As an example of the application of the deformation quantization formalism, two-point correlation functions for the bosonic string are computed in section 5. Finally in section 6 we give our final comments.

2. Overview of classical strings

In this section we give a brief overview of classical bosonic string theory. Our aim is not to provide an extensive review of the theory, but to briefly recall the notation and preliminaries, which will be strictly needed in the following sections (for further details see [1–4]). To perform the deformation quantization we consider the string world-sheet Σ embedded into the D-dimensional spacetime M of Lorentzian metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, $\mu, \nu = 0, 1, \ldots, D-1$. This embedding is defined by $X^\mu = X^\mu(\sigma^a)$, $a = 0, 1$, where σ^a are the coordinates on Σ.

Let g_{ab} be a Riemannian metric of Lorentzian signature $(-,+)$ on Σ. The dynamics of the scalar fields X^μ is described by the Polyakov action:

$$S_P = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu \tag{2.1}$$

where T denotes the string tension. In the conformal gauge ($g_{ab} = \eta_{ab}$) the equations of motion are

$$\partial_a(\eta^{ab}\partial_b X^\mu) = 0 \tag{2.2}$$

while the constraints ($T_{ab} = 0$) are given by

$$\dot{X}^\mu \dot{X}_\mu = 0 \qquad \dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu = 0 \tag{2.3}$$

where $\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial\sigma^0}$ and $X'^\mu \equiv \frac{\partial X^\mu}{\partial\sigma^1}$.

2.1. Closed strings

The general solution of equation (2.2) satisfying the closed string boundary condition $X^\mu(\tau, 0) = X^\mu(\tau, \pi)$ can be written in the form of the following series:

$$X^\mu(\tau,\sigma) = x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{1}{\sqrt{2\pi T}} \times \sum_{n\neq 0}\sqrt{\frac{\hbar}{2|n|}}\{a_n^\mu \exp(2\mathrm{i}(n\sigma - |n|\tau)) + a_n^{\mu *}\exp(-2\mathrm{i}(n\sigma - |n|\tau))\} \tag{2.4}$$

where x^μ and p^μ are real variables representing the centre-of-mass coordinates of the phase space. The conjugate momentum Π^μ of X^μ is, as usual, defined by

$$\Pi^\mu(\tau,\sigma) = \frac{1}{\pi}p^\mu + \mathrm{i}\sqrt{\frac{T}{2\pi}}\sum_{n\neq 0}\sqrt{2\hbar|n|}\{a_n^{\mu *}\exp(-2\mathrm{i}(n\sigma - |n|\tau)) - a_n^\mu \exp(2\mathrm{i}(n\sigma - |n|\tau))\}. \tag{2.5}$$

These solutions (X^μ, Π^μ) satisfy the standard Poisson brackets:

$$\begin{aligned}&\{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma')\\ &\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = 0 = \{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\}\end{aligned} \tag{2.6}$$

and lead to the following Poisson brackets for x^μ, p^μ, a_n^μ and $a_n^{\mu *}$:

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \qquad \{a_m^\mu, a_n^{\nu *}\} = -\frac{\mathrm{i}}{\hbar}\delta_{mn}\eta^{\mu\nu} \tag{2.7}$$

with the remaining independent Poisson brackets being zero.

The bosonic strings formalism is usually expressed in terms of the α variables $\alpha_n^\mu = -\mathrm{i}\sqrt{\hbar n}a_n^\mu$, $\tilde{\alpha}_n^\mu = -\mathrm{i}\sqrt{\hbar n}a_{-n}^\mu$, $\alpha_{-n}^\mu = \alpha_n^{\mu *} = \mathrm{i}\sqrt{\hbar n}a_n^{\mu *}$ and $\tilde{\alpha}_{-n}^\mu = \tilde{\alpha}_n^{\mu *} = \mathrm{i}\sqrt{\hbar n}a_{-n}^{\mu *}$ (for $n > 0$). Thus X^μ and Π^μ are reexpressed as

$$\begin{aligned}X^\mu(\tau,\sigma) = x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{\mathrm{i}}{2\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\{&\alpha_n^\mu\exp(-2\mathrm{i}n(\tau-\sigma))\\ &+\tilde{\alpha}_n^\mu\exp(-2\mathrm{i}n(\tau+\sigma))\}\end{aligned} \tag{2.8}$$

$$\Pi^\mu(\tau,\sigma) = \frac{1}{\pi}p^\mu + \sqrt{\frac{T}{\pi}}\sum_{n\neq 0}\{\alpha_n^\mu\exp(-2\mathrm{i}n(\tau-\sigma)) + \tilde{\alpha}_n^\mu\exp(-2\mathrm{i}n(\tau+\sigma))\} \tag{2.9}$$

while the Poisson brackets (2.7) are

$$\begin{aligned}&\{x^\mu, p^\nu\} = \eta^{\mu\nu}\\ &\{\alpha_m^\mu, \alpha_n^\nu\} = -\mathrm{i}m\delta_{m+n,0}\eta^{\mu\nu} \qquad \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = -\mathrm{i}m\delta_{m+n,0}\eta^{\mu\nu}\end{aligned} \tag{2.10}$$

for all $m, n \neq 0$.

In the *light-cone gauge* the constraint equations (2.3) can be easily solved and then eliminated. This gauge will be crucial for the deformation quantization of the bosonic string in order to identify the relevant phase space when implementing this quantization.

First, introduce the light-cone (null) coordinates $X^\pm := \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$, and the remaining coordinates X^j, $j = 1, \ldots, D-2$ are left as before. As $X^+(\tau,\sigma)$ satisfies the wave equation (2.2) one can choose the coordinate τ in such a manner that $X^+(\tau,\sigma) = \frac{1}{\pi T}p^+\tau$. In this gauge we can solve the constraint equations in the sense that p^-, α^- and $\tilde{\alpha}^-$ are defined by p^+, p^j, α_n^j and $\tilde{\alpha}_n^j$. Thus the (independent) dynamical variables of the string are x^-, p^+, x^j, p^j, α_n^j and $\tilde{\alpha}_n^j$ for $n \neq 0$ or, equivalently, x^-, p^+, X^j and Π^j. For the Poisson bracket for x^- and p^+ we have $\{x^-, p^+\} = -1$.

In the light-cone gauge the square mass $M^2 = -p^\mu p_\mu$ now takes the form

$$M^2 = 4\pi T\sum_{j=1}^{D-2}\sum_{n\neq 0}\alpha_n^j\alpha_{-n}^j = 4\pi T\sum_{j=1}^{D-2}\sum_{n\neq 0}\tilde{\alpha}_n^j\tilde{\alpha}_{-n}^j. \tag{2.11}$$

Then the Hamiltonian $H = \frac{T}{2}\int_0^\pi \mathrm{d}\sigma\{\sum_{j=1}^{D-2}((\frac{\Pi^j}{T})^2 + (X'^j)^2)\}$ is

$$H = \frac{\sum_{j=1}^{D-2}(p^j)^2}{2\pi T} + 2\hbar\sum_{j=1}^{D-2}\sum_{n\neq 0}|n|a_n^{j*}a_n^j = \frac{p^+p^-}{\pi T}. \tag{2.12}$$

Now analogously as in the case of classical fields [18, 20] we introduce the oscillator variables Q_n^j and P_n^j, $n \neq 0$, as follows:

$$Q_n^j(\tau) := \sqrt{\frac{\hbar}{4|n|}}(a_n^j(\tau) + a_n^{j*}(\tau)) \qquad P_n^j(\tau) := \mathrm{i}\sqrt{\hbar|n|}(a_n^{j*}(\tau) - a_n^j(\tau)) \tag{2.13}$$

where $a_n^j(\tau) := a_n^j \exp(-2\mathrm{i}|n|\tau)$.

By relations (2.7) the Poisson brackets of the oscillator variables are given by

$$\begin{aligned} &\{Q_m^j(\tau), P_n^k(\tau)\} = \delta^{jk}\delta_{mn} \\ &\{Q_m^j(\tau), Q_n^k(\tau)\} = 0 = \{P_m^j(\tau), P_n^k(\tau)\}. \end{aligned} \tag{2.14}$$

From equation (2.13) we quickly find

$$a_n^j(\tau) = \sqrt{\frac{|n|}{\hbar}}\left(Q_n^j(\tau) + \frac{\mathrm{i}}{2|n|}P_n^j(\tau)\right). \tag{2.15}$$

Straightforward calculations show that equations (2.4) and (2.5) give

$$a_n^j(\tau) = \frac{1}{2\sqrt{\pi\hbar|n|}}\int_0^\pi \mathrm{d}\sigma\left(2|n|\sqrt{T}X^j(\tau,\sigma) + \frac{\mathrm{i}}{\sqrt{T}}\Pi^j(\tau,\sigma)\right)\exp(-2\mathrm{i}n\sigma). \tag{2.16}$$

Substituting equation (2.16) into (2.13) one gets

$$\begin{aligned} Q_n^j(\tau) = \frac{1}{\sqrt{\pi}}\int_0^\pi \mathrm{d}\sigma\bigg(&\sqrt{T}X^j(\sigma)\cos(2n\sigma + 2|n|\tau) \\ &+ \frac{1}{2|n|\sqrt{T}}\Pi^j(\sigma)\sin(2n\sigma + 2|n|\tau)\bigg) \\ P_n^j(\tau) = \frac{1}{\sqrt{\pi}}\int_0^\pi \mathrm{d}\sigma\bigg(&-2|n|\sqrt{T}X^j(\sigma)\sin(2n\sigma + 2|n|\tau) \\ &+ \frac{1}{\sqrt{T}}\Pi^j(\sigma)\cos(2n\sigma + 2|n|\tau)\bigg) \end{aligned} \tag{2.17}$$

where $X^j(\sigma) \equiv X^j(0,\sigma)$ and $\Pi^j(\sigma) \equiv \Pi^j(0,\sigma)$.

Inserting equation (2.15) into (2.4) and (2.5) the inverse equation can be easily obtained:

$$\begin{aligned} X^j(\tau,\sigma) = x^j + \frac{1}{\pi T}p^j\tau + \frac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\bigg(&Q_n^j\cos(2n\sigma - 2|n|\tau) \\ &- \frac{1}{2|n|}P_n^j\sin(2n\sigma - 2|n|\tau)\bigg) \\ \Pi^j(\tau,\sigma) = \frac{1}{\pi}p^j + \sqrt{\frac{T}{\pi}}\sum_{n\neq 0}&(2|n|Q_n^j\sin(2n\sigma - 2|n|\tau) + P_n^j\cos(2n\sigma - 2|n|\tau)) \end{aligned} \tag{2.18}$$

where $Q_n^j \equiv Q_n^j(0)$ and $P_n^j \equiv P_n^j(0)$.

Observe also that from equation (2.13) one quickly finds that

$$\begin{aligned} Q_n^j(\tau) &= Q_n^j\cos(2|n|\tau) + \frac{1}{2|n|}P_n^j\sin(2|n|\tau) \\ P_n^j(\tau) &= -2|n|Q_n^j\sin(2|n|\tau) + P_n^j\cos(2|n|\tau). \end{aligned} \tag{2.19}$$

Finally, from equations (2.11), (2.12) and (2.15) the mass squared and the Hamiltonian can be expressed in terms of the (P, Q) variables:

$$M^2 = 4\pi T\sum_{j=1}^{D-2}\sum_{n\neq 0}((P_n^j)^2 + 4n^2(Q_n^j)^2) \tag{2.20}$$

and

$$H = \frac{\sum_{j=1}^{D-2}(p^j)^2}{2\pi T} + \frac{1}{2}\sum_{j=1}^{D-2}\sum_{n\neq 0}((P_n^j)^2 + 4n^2(Q_n^j)^2). \tag{2.21}$$

Thus one can use the (independent) dynamical variables $(x^-, p^+, x^j, p^j, Q_n^j, P_n^j)$ and these variables are canonically related to the variables (x^-, p^+, X^j, Π^j). Straightforward calculations give

$$\begin{aligned}&\{X^j(\tau,\sigma), \Pi^k(\tau,\sigma')\}_{(x,p,Q,P)} := \delta^{jk}\delta(\sigma-\sigma')\\ &\{X^j(\tau,\sigma), X^k(\tau,\sigma')\}_{(x,p,Q,P)} = 0 = \{\Pi^j(\tau,\sigma), \Pi^k(\tau,\sigma')\}_{(x,p,Q,P)}.\end{aligned} \tag{2.22}$$

2.2. *Open strings*

In this case the general solution of equation (2.2) satisfying the open string boundary condition $X'^\mu(\tau,0) = X'^\mu(\tau,\pi) = 0$ can be represented by the series

$$\begin{aligned}X^\mu(\tau,\sigma) &= x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\sqrt{\frac{\hbar}{n}}(a_n^\mu\exp(-\mathrm{i}n\tau) + a_n^{\mu*}\exp(\mathrm{i}n\tau))\cos(n\sigma)\\ &= x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{\mathrm{i}}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\exp(-\mathrm{i}n\tau)\cos(n\sigma).\end{aligned} \tag{2.23}$$

Note that the above boundary condition at $\sigma = 0$ yields $a_n^\mu = a_{-n}^\mu$ for all $n \neq 0$. Here α_n^μ are defined as before and $\tilde{\alpha}_n^\mu$ do not appear as independent variables because $a_n^\mu = a_{-n}^\mu$.

Then

$$\begin{aligned}\Pi^\mu(\tau,\sigma) = T\dot{X}^\mu &= \frac{1}{\pi}p^\mu + \mathrm{i}\sqrt{\frac{T}{\pi}}\sum_{n=1}^{\infty}\sqrt{\hbar n}(a_n^{\mu*}\exp(\mathrm{i}n\tau) - a_n^\mu\exp(-\mathrm{i}n\tau))\cos(n\sigma)\\ &= \frac{1}{\pi}p^\mu + \sqrt{\frac{T}{\pi}}\sum_{n\neq 0}\alpha_n^\mu\exp(-\mathrm{i}n\tau)\cos(n\sigma).\end{aligned} \tag{2.24}$$

In the light-cone gauge the squared mass M^2 is given by

$$M^2 = -p^\mu p_\mu = \pi T\sum_{j=1}^{D-2}\sum_{n\neq 0}\alpha_n^j\alpha_{-n}^j \tag{2.25}$$

while the Hamiltonian is

$$H = \frac{1}{2}\sum_{j=1}^{D-2}\sum_{n=-\infty}^{\infty}\alpha_n^j\alpha_{-n}^j = \frac{\sum_{j=1}^{D-2}(p^j)^2}{2\pi T} + \hbar\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}na_n^ja_n^{j*}. \tag{2.26}$$

Analogously as with the closed string case we introduce new variables Q_n^j and P_n^j, $n = 1, \ldots, \infty$:

$$\begin{aligned}Q_n^j(\tau) &:= \sqrt{\frac{\hbar}{2n}}(a_n^j(\tau) + a_n^{j*}(\tau)) = \frac{\mathrm{i}}{n\sqrt{2}}(\alpha_n^j(\tau) - \alpha_{-n}^j(\tau))\\ P_n^j(\tau) &:= \mathrm{i}\sqrt{\frac{\hbar n}{2}}(a_n^{j*}(\tau) - a_n^j(\tau)) = \frac{1}{\sqrt{2}}(\alpha_n^j(\tau) + \alpha_{-n}^j(\tau))\end{aligned} \tag{2.27}$$

where $a_n^j(\tau) := a_n^j\exp(-\mathrm{i}n\tau)$, $n \in \mathbb{Z}_+$. $Q_n^j(\tau)$ and $P_n^j(\tau)$ fulfil the Poisson bracket formulae (2.14). Then

$$a_n^j(\tau) = \sqrt{\frac{n}{2\hbar}}\left(Q_n^j(\tau) + \frac{\mathrm{i}}{n}P_n^j(\tau)\right) \qquad n \in \mathbb{Z}_+. \tag{2.28}$$

Inserting equation (2.28) into (2.23) and (2.24) one gets

$$
\begin{aligned}
X^j(\tau,\sigma) &= x^j + \frac{1}{\pi T}p^j\tau + \sqrt{\frac{2}{\pi T}}\sum_{n=1}^{\infty}\left(Q_n^j\cos(n\tau) + \frac{1}{n}P_n^j\sin(n\tau)\right)\cos(n\sigma)\\
\Pi^j(\tau,\sigma) &= \frac{1}{\pi}p^j + \sqrt{\frac{2T}{\pi}}\sum_{n=1}^{\infty}(-nQ_n^j\sin(n\tau) + P_n^j\cos(n\tau))\cos(n\sigma)
\end{aligned}
\tag{2.29}
$$

where, as before, $Q_n^j \equiv Q_n^j(0)$ and $P_n^j \equiv P_n^j(0)$.

From equations (2.23) and (2.24) we find

$$
a_n^j(\tau) = \frac{1}{\sqrt{\pi\hbar n}}\int_0^{\pi}\mathrm{d}\sigma\left(n\sqrt{T}X^j(\tau,\sigma) + \frac{\mathrm{i}}{\sqrt{T}}\Pi^j(\tau,\sigma)\right)\cos(n\sigma). \tag{2.30}
$$

Substituting equation (2.30) into (2.27) one quickly obtains

$$
\begin{aligned}
Q_n^j(\tau) &= \sqrt{\frac{2}{\pi}}\int_0^{\pi}\mathrm{d}\sigma\left(\sqrt{T}X^j(\sigma)\cos(n\tau) + \frac{1}{n\sqrt{T}}\Pi^j(\sigma)\sin(n\tau)\right)\cos(n\sigma)\\
P_n^j(\tau) &= \sqrt{\frac{2}{\pi}}\int_0^{\pi}\mathrm{d}\sigma\left(-n\sqrt{T}X^j(\sigma)\sin(n\tau) + \frac{1}{\sqrt{T}}\Pi^j(\sigma)\cos(n\tau)\right)\cos(n\sigma).
\end{aligned}
\tag{2.31}
$$

From equation (2.27) we have

$$
\begin{aligned}
Q_n^j(\tau) &= Q_n^j\cos(n\tau) + \frac{1}{n}P_n^j\sin(n\tau)\\
P_n^j(\tau) &= -nQ_n^j\sin(n\tau) + P_n^j\cos(n\tau)
\end{aligned}
\tag{2.32}
$$

(compare with equation (2.19)).

Finally, M^2 and H in terms of oscillator variables are given by

$$
M^2 = \pi T\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}((P_n^j)^2 + n^2(Q_n^j)^2) \tag{2.33}
$$

and

$$
H = \frac{\sum_{j=1}^{D-2}(p^j)^2}{2\pi T} + \frac{1}{2}\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}((P_n^j)^2 + n^2(Q_n^j)^2). \tag{2.34}
$$

As before we can use the (independent) dynamical variables $(x^-, p^+, x^j, p^j, Q_n^j, P_n^j)$ and they are canonically related to (x^-, p^+, X^j, Π^j). Observe that, in the present case, $n \in \mathbb{Z}_+$.

3. Deformation quantization of the bosonic string

In this section we are going to use the well known machinery of deformation quantization [21–32] for the case of bosonic strings. We show that the bosonic string in the light-cone gauge can be quantized by a deformation of the classical theory. Here we also show that the Weyl–Wigner–Moyal formalism can be carried over to string theory. This constitutes a new application of the deformation quantization formalism.

3.1. Closed strings

According to the overview of section 2 the phase space $\mathcal{Z}$ of a closed string can be understood as the Cartesian product $\mathcal{Z} = \mathbb{R}^2 \times \mathbb{R}^{2(D-2)} \times \mathbb{R}^{2\infty}$ endowed with the following symplectic form:

$$
\omega = \mathrm{d}p_- \wedge \mathrm{d}x^- + \sum_{j=1}^{D-2}\left(\mathrm{d}p_j \wedge \mathrm{d}x^j + \sum_{n\neq 0}\mathrm{d}P_{jn}\wedge \mathrm{d}Q_n^j\right) \tag{3.1}
$$

where $p_- = -p^+$, $p_j = p^j$ and $P_{jn} = P_n^j$.

Equivalently, one can consider $\mathcal{Z}$ to be $\mathcal{Z} = \mathbb{R}^2 \times \Gamma$ where Γ is the set $\Gamma = \{(X^j(\sigma), \Pi_j(\sigma))_{j=1,\dots,D-2}\}$, with $X^j(\sigma)$ and $\Pi^j(\sigma) = \Pi_j(\sigma)$ being arbitrary real functions of $\sigma \in [0, \pi]$ satisfying the boundary conditions $X^j(0) = X^j(\pi)$ and $\Pi^j(0) = \Pi^j(\pi)$. The symplectic form now has the functional form

$$\omega = \mathrm{d}p_- \wedge \mathrm{d}x^- + \sum_{j=1}^{D-2} \int_0^\pi \mathrm{d}\sigma\, \delta\Pi_j(\sigma) \wedge \delta X^j(\sigma). \tag{3.2}$$

Let $\hat{x}^-$, $\hat{p}^+ = -\hat{p}_-$, $\hat{X}^j$ and $\hat{\Pi}^j$ be the field operators satisfying

$$\begin{aligned} &\hat{x}^-|x^-\rangle = x^-|x^-\rangle \qquad && \hat{p}^+|p^+\rangle = p^+|p^+\rangle \\ &\hat{X}^j(\sigma)|X^j\rangle = X^j(\sigma)|X^j\rangle \qquad && \hat{\Pi}^j(\sigma)|\Pi^j\rangle = \Pi^j(\sigma)|\Pi^j\rangle \\ &[\hat{X}^j(\sigma), \hat{\Pi}^k(\sigma')] = \mathrm{i}\hbar\delta^{jk}\delta(\sigma - \sigma') \qquad && [\hat{x}^-, \hat{p}^+] = -\mathrm{i}\hbar. \end{aligned} \tag{3.3}$$

As usual, the Fock space can be constructed from the centre-of-mass variables plus the oscillator variables as follows:

$$|x^-, X\rangle := |x^-\rangle \otimes \left(\bigotimes_{j=1}^{D-2} |X^j\rangle\right) \qquad |p^+, \Pi\rangle := |p^+\rangle \otimes \left(\bigotimes_{j=1}^{D-2} |\Pi^j\rangle\right)$$

while the measures of the functional integrals are given by

$$\mathcal{D}X = \prod_\sigma \mathrm{d}X^1(\sigma), \dots, \mathrm{d}X^{D-2}(\sigma) \qquad \text{and} \qquad \mathcal{D}\Pi = \prod_\sigma \mathrm{d}\Pi^1(\sigma), \dots, \mathrm{d}\Pi^{D-2}(\sigma). \tag{3.4}$$

With these definitions we fix the normalization of these states as follows:

$$\begin{aligned} &\int \mathrm{d}x^-\, \mathcal{D}X |x^-, X\rangle\langle x^-, X| = \hat{1} \qquad \text{and} \\ &\int \mathrm{d}\left(\frac{p^+}{2\pi\hbar}\right) \mathcal{D}\left(\frac{\Pi}{2\pi\hbar}\right) |p^+, \Pi\rangle\langle p^+, \Pi| = \hat{1}. \end{aligned} \tag{3.5}$$

Let $F = F[x^-, X, p^+, \Pi]$ be a functional on the phase space $\mathcal{Z}$. Then, according to the Weyl rule we assign the following operator $\hat{F}$ corresponding to F:

$$\hat{F} = W(F) = \int \frac{\mathrm{d}x^-\,\mathrm{d}p^+}{2\pi\hbar} \mathcal{D}X \mathcal{D}\left(\frac{\Pi}{2\pi\hbar}\right) F[x^-, X, p^+, \Pi] \hat{\Omega}[x^-, X, p^+, \Pi] \tag{3.6}$$

where $\hat{\Omega}[x^-, X, p^+, \Pi]$ is the *Stratonovich–Weyl (SW) quantizer*:

$$\begin{aligned} \hat{\Omega}[x^-, X, p^+, \Pi] &= \int \mathrm{d}\xi^-\, \mathcal{D}\xi \exp\left\{-\frac{\mathrm{i}}{\hbar}\left(-\xi^- p^+ + \int_0^\pi \mathrm{d}\sigma\, \xi(\sigma)\cdot\Pi(\sigma)\right)\right\} \\ &\quad \times \left| x^- - \frac{\xi^-}{2}, X - \frac{\xi}{2}\right\rangle\left\langle X + \frac{\xi}{2}, x^- + \frac{\xi^-}{2}\right| \\ &= \int \mathrm{d}\left(\frac{\eta^+}{2\pi\hbar}\right)\mathcal{D}\left(\frac{\eta}{2\pi\hbar}\right) \exp\left\{-\frac{\mathrm{i}}{\hbar}\left(-x^-\eta^+ + \int_0^\pi \mathrm{d}\sigma\, \eta(\sigma)\cdot X(\sigma)\right)\right\} \\ &\quad \times \left| p^+ + \frac{\eta^+}{2}, \Pi + \frac{\eta}{2}\right\rangle\left\langle \Pi - \frac{\eta}{2}, p^+ - \frac{\eta^+}{2}\right| \end{aligned} \tag{3.7}$$

with the obvious notation $\xi(\sigma)\cdot\Pi(\sigma) \equiv \sum_{j=1}^{D-2} \xi^j(\sigma)\Pi^j(\sigma)$ and $\eta(\sigma)\cdot X(\sigma) \equiv \sum_{j=1}^{D-2} \eta^j(\sigma) X^j(\sigma)$.

The SW quantizer has the important properties

$$(\hat{\Omega}[x^-, X, p^+, \Pi])^\dagger = \hat{\Omega}[x^-, X, p^+, \Pi] \tag{3.8}$$

$$\mathrm{Tr}(\hat{\Omega}[x^-, X, p^+, \Pi]) = 1 \tag{3.9}$$

$$\begin{aligned}\mathrm{Tr}(\hat{\Omega}[x^-, X, p^+, \Pi]\hat{\Omega}['x^-, 'X, 'p^+, '\Pi])\\ = \delta(x^- - 'x^-)\delta\left(\frac{p^+ - 'p^+}{2\pi\hbar}\right)\delta[X - 'X]\delta\left[\frac{\Pi - '\Pi}{2\pi\hbar}\right].\end{aligned} \tag{3.10}$$

Multiplying equation (3.6) by $\hat{\Omega}[x^-, X, p^+, \Pi]$ and taking the trace one has

$$W^{-1}(\hat{F}) = F[x^-, X, p^+, \Pi] = \mathrm{Tr}(\hat{\Omega}[x^-, X, p^+, \Pi]\hat{F}). \tag{3.11}$$

This enables us to solve the following problem. Let $F_1 = F_1[x^-, X, p^+, \Pi]$ and $F_2 = F_2[x^-, X, p^+, \Pi]$ be functionals defined on the phase space $\mathcal{Z}$ and let $\hat{F}_1 = W(F_1)$ and $\hat{F}_2 = W(F_2)$ be their corresponding operators. The problem is what functional on $\mathcal{Z}$ corresponds to the product $\hat{F}_1\hat{F}_2$. This functional is denoted by $F_1 * F_2$ and it is called the *Moyal $*$ product of F_1 and F_2*.

By equation (3.11) one gets

$$(F_1 * F_2)[x^-, X, p^+, \Pi] := W^{-1}(\hat{F}_1\hat{F}_2) = \mathrm{Tr}(\hat{\Omega}[x^-, X, p^+, \Pi]\hat{F}_1\hat{F}_2). \tag{3.12}$$

Substituting equation (3.6) into (3.12), then using (3.7) and perfoming straightforward but tedious manipulations (see, e.g., [31]) we finally obtain

$$\begin{aligned}(F_1 * F_2)[x^-, X, p^+, \Pi] &= F_1[x^-, X, p^+, \Pi]\exp\left\{\frac{\mathrm{i}\hbar}{2}\overleftrightarrow{\mathcal{P}}\right\}F_2[x^-, X, p^+, \Pi]\\ \overleftrightarrow{\mathcal{P}} &:= \left(\frac{\overleftarrow{\partial}}{\partial p^+}\frac{\overrightarrow{\partial}}{\partial x^-} - \frac{\overleftarrow{\partial}}{\partial x^-}\frac{\overrightarrow{\partial}}{\partial p^+}\right) + \sum_{j=1}^{D-2}\int_0^\pi \mathrm{d}\sigma\left(\frac{\overleftarrow{\delta}}{\delta X^j(\sigma)}\frac{\overrightarrow{\delta}}{\delta \Pi^j(\sigma)} - \frac{\overleftarrow{\delta}}{\delta \Pi^j(\sigma)}\frac{\overrightarrow{\delta}}{\delta X^j(\sigma)}\right).\end{aligned} \tag{3.13}$$

Now it is an easy matter to define the Wigner functional. Assume $\hat{\rho}$ to be the density operator of the quantum state of a bosonic string. Then, according to the general formula (3.11) the functional $\rho[x^-, X, p^+, \Pi]$ corresponding to $\hat{\rho}$ is (use also (3.7))

$$\begin{aligned}\rho[x^-, X, p^+, \Pi] &= W^{-1}(\hat{\rho}) = \mathrm{Tr}(\hat{\Omega}[x^-, X, p^+, \Pi]\hat{\rho})\\ &= \int \mathrm{d}\xi^-\mathcal{D}\xi\exp\left\{-\frac{\mathrm{i}}{\hbar}\left(-\xi^- p^+ + \int_0^\pi \mathrm{d}\sigma\,\xi(\sigma)\cdot\Pi(\sigma)\right)\right\}\\ &\quad\times\left\langle X + \frac{\xi}{2}, x^- + \frac{\xi^-}{2}\middle|\hat{\rho}\middle|x^- - \frac{\xi^-}{2}, X - \frac{\xi}{2}\right\rangle.\end{aligned} \tag{3.14}$$

Then the *Wigner functional* $\rho_w[x^-, X, p^+, \Pi]$ is defined by a simple modification of equation (3.14), namely

$$\begin{aligned}\rho_w[x^-, X, p^+, \Pi] &:= \int \mathrm{d}\left(\frac{\xi^-}{2\pi\hbar}\right)\mathcal{D}\left(\frac{\xi}{2\pi\hbar}\right)\exp\left\{-\frac{\mathrm{i}}{\hbar}\left(-\xi^- p^+ + \int_0^\pi \mathrm{d}\sigma\,\xi(\sigma)\cdot\Pi(\sigma)\right)\right\}\\ &\quad\times\left\langle X + \frac{\xi}{2}, x^- + \frac{\xi^-}{2}\middle|\hat{\rho}\middle|x^- - \frac{\xi^-}{2}, X - \frac{\xi}{2}\right\rangle.\end{aligned} \tag{3.15}$$

In particular, for the pure state $\hat{\rho} = |\Psi\rangle\langle\Psi|$ we get

$$\begin{aligned}\rho_w[x^-, X, p^+, \Pi] &= \int \mathrm{d}\left(\frac{\xi^-}{2\pi\hbar}\right)\mathcal{D}\left(\frac{\xi}{2\pi\hbar}\right)\exp\left\{-\frac{\mathrm{i}}{\hbar}\left(-\xi^- p^+ + \int_0^\pi \mathrm{d}\sigma\,\xi(\sigma)\cdot\Pi(\sigma)\right)\right\}\\ &\quad\times\Psi^*\left[x^- - \frac{\xi^-}{2}, X - \frac{\xi}{2}\right]\Psi\left[x^- + \frac{\xi^-}{2}, X + \frac{\xi}{2}\right]\end{aligned} \tag{3.16}$$

where $\Psi[x^-, X]$ stands for $|\Psi\rangle$ in the Schrödinger representation.

As will become clear very soon some calculations simplify when the variables $(x^-, p^+, x^j, p^j, Q_n^j, P_n^j)$ are used. In terms of these variables one has (in the obvious notation)

$$\begin{aligned}\hat{\Omega}(x^-, Q, p^+, P) &= \int \mathrm{d}\xi^- \,\mathrm{d}\xi \exp\left\{-\frac{\mathrm{i}}{\hbar}(-\xi^- p^+ + \xi\cdot P)\right\} \\ &\quad\times \left| x^- - \frac{\xi^-}{2}, Q - \frac{\xi}{2}\right\rangle \left\langle Q + \frac{\xi}{2}, x^- + \frac{\xi^-}{2}\right| \\ &= \int \mathrm{d}\left(\frac{\eta^+}{2\pi\hbar}\right) \mathrm{d}\left(\frac{\eta}{2\pi\hbar}\right) \exp\left\{-\frac{\mathrm{i}}{\hbar}(-x^-\eta^+ + \eta\cdot Q)\right\} \\ &\quad\times \left| p^+ + \frac{\eta^+}{2}, P + \frac{\eta}{2}\right\rangle \left\langle P - \frac{\eta}{2}, p^+ - \frac{\eta^+}{2}\right| \end{aligned} \tag{3.17}$$

where $\mathrm{d}\xi \equiv \prod_{n\in\mathbb{Z}} \mathrm{d}\xi_n^1, \ldots, \mathrm{d}\xi_n^{D-2}$, $\mathrm{d}(\frac{\eta}{2\pi\hbar}) \equiv \prod_{n\in\mathbb{Z}} \mathrm{d}(\frac{\eta_n^1}{2\pi\hbar}), \ldots, \mathrm{d}(\frac{\eta_n^{D-2}}{2\pi\hbar})$, $\xi\cdot P \equiv \sum_{j=1}^{D-2}\sum_{n=-\infty}^{\infty} \xi_n^j P_n^j$, $\eta\cdot Q \equiv \sum_{j=1}^{D-2}\sum_{n=-\infty}^{\infty} \eta_n^j Q_n^j$, $P_0^j \equiv p^j$ and $Q_0^j \equiv x^j$.

Then the Moyal $*$ product in terms of these variables is

$$\begin{aligned}(F_1 * F_2)(x^-, Q, p^+, P) &= F_1(x^-, Q, p^+, P) \exp\left\{\frac{\mathrm{i}\hbar}{2}\overleftrightarrow{\mathcal{P}}\right\} F_2(x^-, Q, p^+, P) \\ \overleftrightarrow{\mathcal{P}} &:= \left(\frac{\overleftarrow{\partial}}{\partial p^+}\frac{\overrightarrow{\partial}}{\partial x^-} - \frac{\overleftarrow{\partial}}{\partial x^-}\frac{\overrightarrow{\partial}}{\partial p^+}\right) + \sum_{j=1}^{D-2}\sum_{n=-\infty}^{\infty}\left(\frac{\overleftarrow{\partial}}{\partial Q_n^j}\frac{\overrightarrow{\partial}}{\partial P_n^j} - \frac{\overleftarrow{\partial}}{\partial P_n^j}\frac{\overrightarrow{\partial}}{\partial Q_n^j}\right).\end{aligned} \tag{3.18}$$

We can also express the Moyal $*$ product in terms of a_n^j and a_n^{j*} or α_n^j and $\tilde{\alpha}_n^j$:

$$\begin{aligned} * &= \exp\left\{\frac{\mathrm{i}\hbar}{2}\overleftrightarrow{\mathcal{P}}\right\} \\ &= \exp\left\{\frac{\mathrm{i}\hbar}{2}\left[\left(\frac{\overleftarrow{\partial}}{\partial p^+}\frac{\overrightarrow{\partial}}{\partial x^-} - \frac{\overleftarrow{\partial}}{\partial x^-}\frac{\overrightarrow{\partial}}{\partial p^+}\right) + \sum_{j=1}^{D-2}\left(\frac{\overleftarrow{\partial}}{\partial x^j}\frac{\overrightarrow{\partial}}{\partial p^j} - \frac{\overleftarrow{\partial}}{\partial p^j}\frac{\overrightarrow{\partial}}{\partial x^j}\right)\right]\right\} \\ &\quad\times \exp\left\{\frac{1}{2}\sum_{j=1}^{D-2}\sum_{n\neq 0}\left(\frac{\overleftarrow{\partial}}{\partial a_n^j}\frac{\overrightarrow{\partial}}{\partial a_n^{j*}} - \frac{\overleftarrow{\partial}}{\partial a_n^{j*}}\frac{\overrightarrow{\partial}}{\partial a_n^j}\right)\right\} \\ &= \cdots \exp\left\{\frac{\hbar}{2}\sum_{j=1}^{D-2}\sum_{n\neq 0} n\left(\frac{\overleftarrow{\partial}}{\partial \alpha_n^j}\frac{\overrightarrow{\partial}}{\partial \alpha_{-n}^j} + \frac{\overleftarrow{\partial}}{\partial \tilde{\alpha}_n^j}\frac{\overrightarrow{\partial}}{\partial \tilde{\alpha}_{-n}^j}\right)\right\}. \end{aligned} \tag{3.19}$$

Finally, for the Wigner function one obtains

$$\begin{aligned}\rho_w(x^-, Q, p^+, P) &= \int \mathrm{d}\left(\frac{\xi^-}{2\pi\hbar}\right) \mathrm{d}\left(\frac{\xi}{2\pi\hbar}\right) \exp\left\{-\frac{\mathrm{i}}{\hbar}(-\xi^- p^+ + \xi\cdot P)\right\} \\ &\quad \left\langle Q + \frac{\xi}{2}, x^- + \frac{\xi^-}{2}\right| \hat{\rho} \left| x^- - \frac{\xi^-}{2}, Q - \frac{\xi}{2}\right\rangle \end{aligned} \tag{3.20}$$

and in the case of the pure state $\hat{\rho} = |\Psi\rangle\langle\Psi|$

$$\begin{aligned}\rho_w(x^-, Q, p^+, P) &= \int \mathrm{d}\left(\frac{\xi^-}{2\pi\hbar}\right) \mathrm{d}\left(\frac{\xi}{2\pi\hbar}\right) \exp\left\{-\frac{\mathrm{i}}{\hbar}(-\xi^- p^+ + \xi\cdot P)\right\} \\ &\quad\times \Psi^*\left(x^- - \frac{\xi^-}{2}, Q - \frac{\xi}{2}\right) \Psi\left(x^- + \frac{\xi^-}{2}, Q + \frac{\xi}{2}\right). \end{aligned} \tag{3.21}$$

Given ρ_w one can use equation (3.6) to find the corresponding density operator $\hat{\rho}$:

$$\hat{\rho} = \int \mathrm{d}x^- \,\mathrm{d}p^+ \,\mathrm{d}Q \,\mathrm{d}P \,\rho_w(x^-, Q, p^+, P)\hat{\Omega}(x^-, Q, p^+, P). \tag{3.22}$$

Consequently, the average value $\langle\hat{F}\rangle$ is

$$\begin{aligned}\langle\hat{F}\rangle &= \frac{\mathrm{Tr}(\hat{\rho}\hat{F})}{\mathrm{Tr}\{\hat{\rho}\}}\\ &= \frac{\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_w(x^-,Q,p^+,P)\,\mathrm{Tr}(\hat{\Omega}(x^-,Q,p^+,P)\hat{F})}{\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_w(x^-,Q,p^+,P)}\\ &= \frac{\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_w(x^-,Q,p^+,P)W^{-1}(\hat{F})(x^-,Q,p^+,P)}{\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_w(x^-,Q,p^+,P)}.\end{aligned} \tag{3.23}$$

Assume that $\hat{\rho} = |\Psi\rangle\langle\Psi|$. Substituting this $\hat{\rho}$ into equation (3.22), multiplying from the left by $\langle\tilde{Q},\tilde{x}^-|$ and from the right by $|\tilde{x}^-,\tilde{Q}\rangle$ and employing equation (3.17) one gets

$$|\Psi(\tilde{x}^-,\tilde{Q})|^2 = \int \mathrm{d}p^+\,\mathrm{d}P\,\rho_w(\tilde{x}^-,\tilde{Q},p^+,P). \tag{3.24}$$

Suppose that $\Psi(\tilde{x}^-,\tilde{Q}) \neq 0$. Then inserting $\hat{\rho} = |\Psi\rangle\langle\Psi|$ into (3.22), multiplying from the left by $\langle Q, x^-|$ and from the right by $|\tilde{x}^-,\tilde{Q}\rangle$, using equations (3.17) and (3.24) we easily find the wavefunction $\Psi(x^-,Q)$ in terms of the corresponding Wigner function ρ_w:

$$\begin{aligned}\Psi(x^-,Q) &= \exp\{\mathrm{i}\varphi\}\\ &\quad\times\frac{\int \mathrm{d}p^+\,\mathrm{d}P\,\rho_w(\frac{x^-+\tilde{x}^-}{2},\frac{Q+\tilde{Q}}{2},p^+,P)\exp\{-\frac{\mathrm{i}}{\hbar}(-(x^--\tilde{x}^-)p^+ + (Q-\tilde{Q})\cdot P)\}}{(\int \mathrm{d}p^+\,\mathrm{d}P\,\rho_w(\tilde{x}^-,\tilde{Q},p^+,P))^{1/2}}\end{aligned} \tag{3.25}$$

where φ is an arbitrary real constant.

Of course, in terms of $X^j(\sigma)$ and $\Pi^j(\sigma)$ one has

$$\begin{aligned}\Psi[x^-,X] &= \exp\{\mathrm{i}\varphi\}\bigg(\int \mathrm{d}p^+\,\mathcal{D}\Pi\rho_w\left[\frac{x^-+\tilde{x}^-}{2},\frac{X+\tilde{X}}{2},p^+,\Pi\right]\\ &\quad\times\exp\left\{-\frac{\mathrm{i}}{\hbar}\bigg(-(x^--\tilde{x}^-)p^+ + \int_0^\pi \mathrm{d}\sigma\,(X(\sigma)-\tilde{X}(\sigma))\cdot\Pi(\sigma)\bigg)\right\}\bigg)\\ &\quad\times\left\{\bigg(\int \mathrm{d}p^+\,\mathcal{D}\Pi\rho_w[\tilde{x}^-,\tilde{X},p^+,\Pi]\bigg)^{1/2}\right\}^{-1}\end{aligned} \tag{3.26}$$

where $X(\sigma)\cdot\Pi(\sigma) \equiv \sum_{j=1}^{D-2} X^j(\sigma)\Pi^j(\sigma)$.

The natural question is: when does a real function $\rho_w(x^-,Q,p^+,P)$ represent some quantum state, i.e. it can be considered to be a Wigner function? The necessary and sufficient condition is

$$\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_w(x^-,Q,p^+,P)(f^* * f)(x^-,Q,p^+,P) \geqslant 0 \tag{3.27}$$

for any $f \in C^\infty(\mathcal{Z})[[\hbar]]$, and

$$\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_w(x^-,Q,p^+,P) > 0. \tag{3.28}$$

(See [29, 32].)

3.2. Example: the ground state

The Wigner function ρ_{w0} of the ground state is defined by

$$a_n^j * \rho_{w0} = 0 \qquad p^j * \rho_{w0} = 0 \qquad \text{and} \qquad p^+ * \rho_{w0} = 0 \tag{3.29}$$

for $j = 1, \ldots, D-2$ and $n \neq 0$.

Employing equation (3.19) we have

$$a_n^j \rho_{w0} + \frac{1}{2}\frac{\partial \rho_{w0}}{\partial a_n^{j*}} = 0 \qquad p^j \rho_{w0} = 0 \qquad \text{and} \qquad p^+ \rho_{w0} = 0 \tag{3.30}$$

for $j = 1, \ldots, D-2$ and $n \neq 0$. The general real solution of equation (3.30) also satisfying (3.27) and (3.28) is

$$\rho_{w0} = C \exp\left\{ -2 \sum_{j=1}^{D-2} \sum_{n\neq 0} a_n^j a_n^{j*} \right\} \delta(p^1), \ldots, \delta(p^{D-2})\delta(p^+) \tag{3.31}$$

where $C > 0$. Consequently, in terms of Q_n^j and P_n^j one gets

$$\rho_{w0} = C \exp\left\{ -\frac{1}{2\hbar} \sum_{j=1}^{D-2} \sum_{n\neq 0} \frac{1}{|n|}((P_n^j)^2 + 4n^2 (Q_n^j)^2) \right\} \delta(p^1), \ldots, \delta(p^{D-2})\delta(p^+). \tag{3.32}$$

Observe that ρ_{w0} is defined by equations (3.27)–(3.29) uniquely up to an arbitrary real constant factor $C > 0$. This fact can be interpreted in the deformation quantization formalism as the *uniqueness of the vacuum state.*

Then any higher state can be obtained as an appropriate product of the form

$$(a_{n_1}^{*i_1}, \ldots, a_{n_s}^{*i_s}) * \left\{ C \exp\left(-\frac{1}{2\hbar} \sum_{j=1}^{D-2} \sum_{n\neq 0} \frac{1}{|n|}((P_n^j)^2 + 4n^2 (Q_n^j)^2) \right) \right.$$
$$\left. \delta(p^1 - p_0^1), \ldots, \delta(p^{D-2} - p_0^{D-2})\delta(p^+ - p_0^+) \right\} * (a_{n_s}^{i_s}, \ldots, a_{n_1}^{i_1}) \tag{3.33}$$

(compare with [18]).

An interesting question is when a real function $\rho_w(x^-, Q, p^+, P)$ satisfying equations (3.27) and (3.28) is the Wigner function of a pure state. The answer to this question in the case of a system of particles can be found in a beautiful paper by Tatarskii [27]. In our case the solution is quite similar. To this end write

$$\gamma(x^-, Q, \tilde{x}^-, \tilde{Q}) := \int \mathrm{d}p^+ \, \mathrm{d}P \, \rho_w \left(\frac{x^- + \tilde{x}^-}{2}, \frac{Q + \tilde{Q}}{2}, p^+, P \right)$$
$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar}[-(x^- - \tilde{x}^-)p^+ + (Q - \tilde{Q})P] \right\}. \tag{3.34}$$

From equation (3.25) it follows that, if ρ_w is the Wigner function of the pure state $|\Psi\rangle\langle\Psi|$, then

$$\frac{\partial^2 \ln \gamma(x^-, Q, \tilde{x}^-, \tilde{Q})}{\partial x^- \partial \tilde{x}^-} = \frac{\partial^2 \ln \gamma(x^-, Q, \tilde{x}^-, \tilde{Q})}{\partial x^- \partial \tilde{Q}_n^j}$$
$$= \frac{\partial^2 \ln \gamma(x^-, Q, \tilde{x}^-, \tilde{Q})}{\partial Q_n^j \partial \tilde{x}^-} = \frac{\partial^2 \ln \gamma(x^-, Q, \tilde{x}^-, \tilde{Q})}{\partial Q_m^j \partial \tilde{Q}_n^k} = 0 \tag{3.35}$$

for every $j, k = 1, \ldots, D-2$ and $m, n \in \mathbb{Z}$ (we put $x^i \equiv Q_0^j$, $p^j \equiv P_0^j$).

Conversely, let γ satisfy equation (3.35). The general solution of (3.35) is then

$$\gamma(x^-, Q, \tilde{x}^-, \tilde{Q}) = \Psi_1(x^-, Q)\Psi_2(\tilde{x}^-, \tilde{Q}). \tag{3.36}$$

As the function ρ_w is assumed to be real we get from equation (3.34)

$$\gamma^*(x^-, Q, \tilde{x}^-, \tilde{Q}) = \gamma(\tilde{x}^-, \tilde{Q}, x^-, Q). \tag{3.37}$$

Consequently, equation (3.36) has the form

$$\gamma(x^-, Q, \tilde{x}^-, \tilde{Q}) = A\Psi_1(x^-, Q)\Psi_1^*(\tilde{x}^-, \tilde{Q}) \tag{3.38}$$

where, by the assumption (3.28), A is a positive real constant. Finally, defining $\Psi := \sqrt{A}\Psi_1(x^-, Q)$ one obtains

$$\gamma(x^-, Q, \tilde{x}^-, \tilde{Q}) = \Psi(x^-, Q)\Psi^*(\tilde{x}^-, \tilde{Q}). \tag{3.39}$$

Substituting $x^- \mapsto x^- + \frac{\xi^-}{2}$, $Q \mapsto Q + \frac{\xi}{2}$, $\tilde{x}^- \mapsto x^- - \frac{\xi^-}{2}$, $\tilde{Q} \mapsto Q - \frac{\xi^-}{2}$, multiplying both sides by $\exp\{-\frac{i}{\hbar}(-\xi^- p^+ + \xi \cdot P)\}$ and integrating with respect to $\mathrm{d}(\frac{\xi^-}{2\pi\hbar})\,\mathrm{d}(\frac{\xi}{2\pi\hbar})$ we get exactly the relation (3.21). This means that our function ρ_w is the Wigner function of the pure state $\Psi(x^-, Q)$. Thus we arrive at the following theorem.

Theorem 3.1. *A real function $\rho_w(x^-, Q, p^+, P)$ satisfying also the conditions (3.27) and (3.28) is the Wigner function of some pure state if and only if the function $\gamma(x^-, Q, \tilde{x}, \tilde{Q})$ defined by (3.34) satisfies equations (3.35).*

In terms of (x^-, X, p^+, Π) variables the conditions (3.35) are

$$\begin{aligned}\frac{\partial^2 \ln\gamma[x^-, X, \tilde{x}^-, \tilde{X}]}{\partial x^- \partial\tilde{x}^-} &= \frac{\partial}{\partial x^-}\frac{\delta \ln\gamma[x^-, X, \tilde{x}^-, \tilde{X}]}{\delta\tilde{X}} \\ &= \frac{\partial}{\partial\tilde{x}^-}\frac{\delta \ln\gamma[x^-, X, \tilde{x}^-, \tilde{X}]}{\delta X} = \frac{\delta^2 \ln\gamma[x^-, X, \tilde{x}^-, \tilde{X}]}{\delta X\delta\tilde{X}} = 0.\end{aligned} \tag{3.40}$$

3.3. Open strings

This is a simple matter to carry over the results of the preceding subsection to the case of open strings. The thing we must take care with is that the subindex n standing at Q_n^j, P_n^j, a_n^j and a_n^{*j} takes the values $n = 1, \ldots, \infty$. (We let also n be zero in the formulae where $Q_0^j \equiv x^j$, $P_0^j \equiv p^j$.) Moreover, we should remember that now the oscillator frequencies $\omega_n = n \in \mathbb{Z}_+$, and not $2|n|$ as before, and also that $\tilde{\alpha}_n$, $\tilde{\alpha}_{-n}$ do not appear. Thus in the present case one gets

$$\begin{aligned} * &= \exp\left\{\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right\} \\ &= \exp\left\{\frac{i\hbar}{2}\left[\left(\frac{\overleftarrow{\partial}}{\partial p^+}\frac{\overrightarrow{\partial}}{\partial x^-} - \frac{\overleftarrow{\partial}}{\partial x^-}\frac{\overrightarrow{\partial}}{\partial p^+}\right) + \sum_{j=1}^{D-2}\left(\frac{\overleftarrow{\partial}}{\partial x^j}\frac{\overrightarrow{\partial}}{\partial p^j} - \frac{\overleftarrow{\partial}}{\partial p^j}\frac{\overrightarrow{\partial}}{\partial x^j}\right)\right]\right\} \\ &\quad \times \exp\left\{\frac{1}{2}\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}\left(\frac{\overleftarrow{\partial}}{\partial a_n^j}\frac{\overrightarrow{\partial}}{\partial a_n^{*j}} - \frac{\overleftarrow{\partial}}{\partial a_n^{*j}}\frac{\overrightarrow{\partial}}{\partial a_n^j}\right)\right\} \\ &= \cdots \exp\left\{\frac{\hbar}{2}\sum_{j=1}^{D-2}\sum_{n\neq 0} n\frac{\overleftarrow{\partial}}{\partial\alpha_n^j}\frac{\overrightarrow{\partial}}{\partial\alpha_{-n}^j}\right\}. \end{aligned} \tag{3.41}$$

Then the Wigner function of the ground state is now

$$\begin{aligned}\rho_{w0} &= C\exp\left\{-2\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}a_n^j a_n^{*j}\right\}\delta(p^1), \ldots, \delta(p^{D-2})\delta(p^+) \\ &= C\exp\left\{-\frac{1}{\hbar}\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}\frac{1}{n}\left((P_n^j)^2 + n^2(Q_n^j)^2\right)\right\}\delta(p^1), \ldots, \delta(p^{D-2})\delta(p^+)\end{aligned} \tag{3.42}$$

with $C > 0$. (Compare with equation (3.21) of [18].)

4. Hamiltonian, square mass, normal ordering and Virasoro algebra

To proceed further with the deformation quantization of bosonic strings we must take into account that not only the Weyl ordering but also the normal ordering should be implemented in this quantization. To this end we first consider the Casimir effect in string theory. Consider the real scalar field $\Phi(\tau,\sigma)$ on the cylindrical spacetime $\mathbb{R}\times \boldsymbol{S}^1$. The circumference of $\boldsymbol{S}^1$ is L. The standard expansion of $\Phi(\tau,\sigma)$ satisfying the boundary conditions $\Phi(\tau,\sigma)=\Phi(\tau,\sigma+nL)$ for all $n\in\mathbb{Z}$ is (compare with equation (2.4))

$$\Phi(\tau,\sigma)=x+\frac{1}{L}p\tau+\frac{1}{\sqrt{L}}\sum_{n\neq 0}\sqrt{\frac{\hbar}{2\omega_n}}\left\{a_n\exp\left(\mathrm{i}\left(\frac{2\pi n}{L}\sigma-\omega_n\tau\right)\right)+a_n^*\right.$$
$$\left.\times\exp\left(-\mathrm{i}\left(\frac{2\pi n}{L}\sigma-\omega_n\tau\right)\right)\right\} \tag{4.1}$$

where $\omega_n=\frac{2\pi|n|}{L}$. The conjugate momentum $\Pi(\tau,\sigma)=\dot{\Phi}(\tau,\sigma)$. Employing the deformation quantization formalism one can show [18] that the expected value of the energy density $\langle T_{00}\rangle(L)$ of the ground state is (the Casimir effect)

$$\langle T_{00}\rangle(L)=-\frac{\pi\hbar}{6L^2}. \tag{4.2}$$

(In terms of the usual quantum field theory see [33–35].)

Consequently, for the total energy $E_0(L)$ of the ground state one gets

$$E_0(L)=L\cdot\langle T_{00}\rangle(L)=-\frac{\pi\hbar}{6L}. \tag{4.3}$$

Consider now the real scalar field $\Phi(\tau,\sigma)$ on $\mathbb{R}\times[0,L]$ but with the boundary conditions $\frac{\partial\Phi(\tau,0)}{\partial\sigma}=0=\frac{\partial\Phi(\tau,L)}{\partial\sigma}$ for all $\tau\in\mathbb{R}$. It is a simple matter to show that now the expansion of $\Phi(\tau,\sigma)$ takes the following form:

$$\Phi(\tau,\sigma)=x+\frac{1}{L}p\tau+\frac{1}{\sqrt{2L}}\sum_{n\neq 0}\sqrt{\frac{\hbar}{2\omega_n}}\left\{a_n\exp\left(\mathrm{i}\left(\frac{\pi n}{L}\sigma-\omega_n\tau\right)\right)+a_n^*\right.$$
$$\left.\times\exp\left(-\mathrm{i}\left(\frac{\pi n}{L}\sigma-\omega_n\tau\right)\right)\right\} \tag{4.4}$$

where $\omega_n=\frac{\pi|n|}{L}$ and $a_n=a_{-n}$. Comparing (4.1) with (4.4) one quickly arrives at the conclusion that the oscillating part in (4.4) is *mutatis mutandi* the same as in (4.1) if, in (4.1), L is changed to $2L$. Hence it follows that in the present case the Casimir effect can be obtained from (4.2) by inserting $2L$ instead of L. Thus we have now

$$\langle T_{00}\rangle(L)=-\frac{\pi\hbar}{24L^2} \tag{4.5}$$

and for the total energy of the ground state

$$E_0(L)=L\cdot\langle T_{00}\rangle(L)=-\frac{\pi\hbar}{24L}. \tag{4.6}$$

We use the above results in the deformation quantization of bosonic strings.

4.1. Closed strings

In this case one can consider $X^j(\tau,\sigma)$, $j=1,\ldots,D-2$, to be $D-2$ real scalar massless fields on the cylindrical worldsheet $\mathbb{R}\times\boldsymbol{S}^1$ with $L=\pi$. Therefore, by equation (4.3), the vacuum energy E_0 is now

$$E_0=-\frac{\hbar(D-2)}{6}=:-4a. \tag{4.7}$$

To obtain this E_0 from the eigenvalue equation we put $\hat{\mathcal{N}}'H$

$$\hat{\mathcal{N}}' := \exp\left\{\sum_{j=1}^{D-2}\sum_{n\neq 0}\left(-\frac{1}{2}+\gamma_n\right)\frac{\partial^2}{\partial a_n^j \partial a_n^{*j}}\right\}$$
$$= \exp\left\{\hbar\sum_{j=1}^{D-2}\sum_{n=1}^{\infty} n\left(\left(-\frac{1}{2}+\gamma_n\right)\frac{\partial^2}{\partial \alpha_n^j \partial \alpha_{-n}^j} + \left(-\frac{1}{2}+\gamma_{-n}\right)\frac{\partial^2}{\partial \tilde{\alpha}_n^j \partial \tilde{\alpha}_{-n}^j}\right)\right\} \tag{4.8}$$

instead of the Hamiltonian H given by equation (2.21). Then

$$\hat{\mathcal{N}}'H * \rho_{w0} = -4a\rho_{w0} \tag{4.9}$$

if

$$\sum_{n\neq 0}|n|\gamma_n = -\frac{D-2}{12}. \tag{4.10}$$

From a previous paper [18] we know that the Weyl image of $\hat{\mathcal{N}}'H$ is

$$W(\hat{\mathcal{N}}'H) =: W(H) : -4a \tag{4.11}$$

where $: W(H) :$ is the normal ordered operator $W(H)$ and it can be written as follows:

$$: W(H) := W(\hat{\mathcal{N}}H)$$
$$\hat{\mathcal{N}} := \exp\left\{-\frac{1}{2}\sum_{j=1}^{D-2}\sum_{n\neq 0}\frac{\partial^2}{\partial a_n^j \partial a_n^{*j}}\right\} = \exp\left\{-\frac{\hbar}{2}\sum_{j=1}^{D-2}\sum_{n=1}^{\infty} n\left(\frac{\partial^2}{\partial \alpha_n^j \partial \alpha_{-n}^j}+\frac{\partial^2}{\partial \tilde{\alpha}_n^j \partial \tilde{\alpha}_{-n}^j}\right)\right\}. \tag{4.12}$$

Analogously for the squared mass given by equation (2.11) we get

$$\hat{\mathcal{N}}'M^2 * \rho_{w0} = -8\pi Ta\rho_{w0}. \tag{4.13}$$

One quickly finds that

$$W(\hat{\mathcal{N}}'M^2) = W(\hat{\mathcal{N}}M^2) - 8\pi Ta =: W(M^2) : -8\pi Ta. \tag{4.14}$$

Given the *normal ordering operator* $\hat{\mathcal{N}}$ and the *generalized normal ordering operator* $\hat{\mathcal{N}}'$ one can define new star products which are cohomologically equivalent to the Moyal $*$ product (see equations (3.13) or (3.18)). These star products will be denoted by $*_{\mathcal{N}}$ and $*_{\mathcal{N}'}$, respectively, and they are given by

$$F_1 *_{\mathcal{N}} F_2 = \hat{\mathcal{N}}^{-1}(\hat{\mathcal{N}}F_1 * \hat{\mathcal{N}}F_2) \qquad F_1 *_{\mathcal{N}'} F_2 = \hat{\mathcal{N}}'^{-1}(\hat{\mathcal{N}}'F_1 * \hat{\mathcal{N}}'F_2). \tag{4.15}$$

Consequently the eigenvalue equations for the Hamiltonian or the squared mass are (compare with equation (4.9) or (4.13))

$$\begin{aligned} H *_{\mathcal{N}'} \rho_w^{\mathcal{N}'} = E\rho_w^{\mathcal{N}'} &\Longrightarrow H *_{\mathcal{N}'} \rho_{w0}^{\mathcal{N}'} = -4a\rho_{w0}^{\mathcal{N}'} \\ M^2 *_{\mathcal{N}'} \rho_w^{\mathcal{N}'} = \mu^2\rho_w^{\mathcal{N}'} &\Longrightarrow M^2 *_{\mathcal{N}'} \rho_{w0}^{\mathcal{N}'} = -8\pi Ta\rho_{w0}^{\mathcal{N}'} \end{aligned} \tag{4.16}$$

where $\rho_w^{\mathcal{N}'} := \hat{\mathcal{N}}^{-1}\rho_w$.

It is an easy matter to show that

$$\begin{aligned} \alpha_{-n} *_{\mathcal{N}} \alpha_n &= \alpha_{-n}\alpha_n & \qquad \alpha_n *_{\mathcal{N}} \alpha_{-n} &= \alpha_n\alpha_{-n} + \hbar n \\ \tilde{\alpha}_{-n} *_{\mathcal{N}} \tilde{\alpha}_n &= \tilde{\alpha}_{-n}\tilde{\alpha}_n & \qquad \tilde{\alpha}_n *_{\mathcal{N}} \tilde{\alpha}_{-n} &= \tilde{\alpha}_n\tilde{\alpha}_{-n} + \hbar n \end{aligned} \tag{4.17}$$

for all $n \in \mathbb{Z}_+$. All other products are the usual products.

Taking into account equation (4.17) one gets

$$\begin{aligned}\{\alpha_m^-, \alpha_n^-\}^{(\mathcal{N})} &= -\mathrm{i}\frac{2\sqrt{\pi T}}{p^+}(m-n)\hat{\mathcal{N}}\alpha_{m+n}^- - \mathrm{i}\hbar\frac{4\pi T}{(p^+)^2}\frac{D-2}{12}m(m^2-1)\delta_{m+n,0}\\ \{\tilde{\alpha}_m^-, \tilde{\alpha}_n^-\}^{(\mathcal{N})} &= -\mathrm{i}\frac{2\sqrt{\pi T}}{p^+}(m-n)\hat{\mathcal{N}}\tilde{\alpha}_{m+n}^- - \mathrm{i}\hbar\frac{4\pi T}{(p^+)^2}\frac{D-2}{12}m(m^2-1)\delta_{m+n,0}\end{aligned} \tag{4.18}$$

where $\{\alpha_m^-, \alpha_n^-\}^{(\mathcal{N})} := \frac{1}{\mathrm{i}\hbar}(\alpha_m^- *_{\mathcal{N}} \alpha_n^- - \alpha_n^- *_{\mathcal{N}} \alpha_m^-)$, etc. Thus we arrive at the *Virasoro algebra with a central extension*.

Remark. (Calculations in equation (4.18) are rather formal. To perform them one must always put α_n on the right to be α_{-m} $m, n \in \mathbb{Z}_+, m \neq n$, and the same for $\tilde{\alpha}_n$ and $\tilde{\alpha}_{-m}$. See the analogous calculations in terms of operator language [4].)

4.2. Open strings

Here we can find the energy of the vacuum state E_0 by substituting $L = \pi$ in (4.6) and taking into account that we deal with $D-2$ scalar fields. Hence

$$E_0 = -\frac{\hbar(D-2)}{24} = -a. \tag{4.19}$$

Now the normal ordering operator $\hat{\mathcal{N}}$ and the generalized normal ordering operator $\hat{\mathcal{N}}'$ are

$$\begin{aligned}\hat{\mathcal{N}} &= \exp\left\{-\frac{1}{2}\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}\frac{\partial^2}{\partial a_n^j \partial a_n^{*j}}\right\} = \exp\left\{-\frac{\hbar}{2}\sum_{j=1}^{D-2}\sum_{n=1}^{\infty} n\frac{\partial^2}{\partial \alpha_n^j \partial \alpha_{-n}^j}\right\}\\ \hat{\mathcal{N}}' &= \exp\left\{\sum_{j=1}^{D-2}\sum_{n=1}^{\infty}\left(\left(-\frac{1}{2}+\beta_n\right)\frac{\partial^2}{\partial a_n^j \partial a_n^{*j}}\right)\right\}\\ &= \exp\left\{\hbar\sum_{j=1}^{D-2}\sum_{n=1}^{\infty} n\left(\left(-\frac{1}{2}+\beta_n\right)\frac{\partial^2}{\partial \alpha_n^j \partial \alpha_{-n}^j}\right)\right\}\end{aligned} \tag{4.20}$$

where

$$\sum_{n=1}^{\infty} n\beta_n = -\frac{D-2}{24}. \tag{4.21}$$

Then

$$\begin{aligned}H *_{\mathcal{N}'} \rho_W^{\mathcal{N}'} &= E\rho_W^{\mathcal{N}'} \Longrightarrow H *_{\mathcal{N}'} \rho_{W0}^{\mathcal{N}'} = -a\rho_{W0}^{\mathcal{N}'}\\ M^2 *_{\mathcal{N}'} \rho_W^{\mathcal{N}'} &= \mu^2\rho_W^{\mathcal{N}'} \Longrightarrow M^2 *_{\mathcal{N}'} \rho_{W0}^{\mathcal{N}'} = -2\pi T a\rho_{W0}^{\mathcal{N}'}.\end{aligned} \tag{4.22}$$

Thus, as before, the ground state is the tachyonic one.

Finally, the *Virasoro algebra with a central extension* is now

$$\{\alpha_m^-, \alpha_n^-\}^{(\mathcal{N})} = -\mathrm{i}\frac{\sqrt{\pi T}}{p^+}(m-n)\hat{\mathcal{N}}\alpha_{m+n}^- - \mathrm{i}\hbar\frac{\pi T}{(p^+)^2}\frac{D-2}{12}m(m^2-1)\delta_{m+n,0}. \tag{4.23}$$

5. Some simple examples: the Wightman functions

Here we are going to present a simple example of calculations within the deformation quantization formalism. Namely, we find the Wightman (Green) functions $\langle X^j(\tau,\sigma) * X^k(\tau',\sigma')\rangle$. By definition (see (3.23))

$$\langle X^j(\tau,\sigma) * X^k(\tau',\sigma')\rangle = \frac{\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_{W0}(x^-, Q, p^+, P)X^j(\tau,\sigma) * X^k(\tau',\sigma')}{\int \mathrm{d}x^-\,\mathrm{d}p^+\,\mathrm{d}Q\,\mathrm{d}P\,\rho_{W0}(x^-, Q, p^+, P)}. \tag{5.1}$$

where ρ_{w0} is given by equation (3.32) (closed string) or (3.42) (open string) and $X^j(\tau, \sigma)$ is given by (2.18) (closed string) or by (2.29) (open string).

5.1. Closed strings

Employing equations (2.19) and (3.18) and perfoming simple integrations (Gaussian integrals) one gets

$$\langle Q_m^j(\tau) * Q_n^k(\tau')\rangle = \delta_{jk}\delta_{mn}\frac{\hbar}{4|m|}\exp(-2\mathrm{i}|m|(\tau-\tau')) \qquad m, n \neq 0. \quad (5.2)$$

Hence, differentiating (5.2) with respect to τ and/or τ' we obtain

$$\begin{aligned}\langle Q_m^j(\tau) * P_n^k(\tau')\rangle &= \delta_{jk}\delta_{mn}\frac{\mathrm{i}\hbar}{2}\exp(-2\mathrm{i}|m|(\tau-\tau')) = -\langle P_m^j(\tau) * Q_n^k(\tau')\rangle \\ \langle P_m^j(\tau) * P_n^k(\tau')\rangle &= \delta_{jk}\delta_{mn}\hbar|m|\exp(-2\mathrm{i}|m|(\tau-\tau')) \qquad m, n \neq 0.\end{aligned} \quad (5.3)$$

Then we have also

$$\begin{aligned}&\langle x^j * p^k\rangle = \delta_{jk}\frac{\mathrm{i}\hbar}{2} = -\langle p^j * x^k\rangle \\ &\langle p^j * p^k\rangle = 0 \qquad \langle x^j * x^k\rangle = \delta_{jk}\langle (x^j)^2\rangle.\end{aligned} \quad (5.4)$$

Using equations (5.2)–(5.4) one easily finds

$$\begin{aligned}\langle X^j(\tau,\sigma) * X^k(\tau',\sigma')\rangle = \delta_{jk}\Bigg\{&\langle x^j x^k\rangle + \frac{\mathrm{i}\hbar}{2\pi T}(\tau'-\tau) + \frac{\hbar}{4\pi T} \\ &\times\sum_{n\neq 0}\frac{\exp(2\mathrm{i}[n(\sigma-\sigma') - |n|(\tau-\tau')])}{|n|}\Bigg\}.\end{aligned} \quad (5.5)$$

Performing the summations in equation (5.5) and removing the part independent of the coordinates $(\tau, \sigma, \tau', \sigma')$ we have

$$\begin{aligned}\langle X^j(\tau,\sigma) * X^k(\tau',\sigma')\rangle \sim \delta_{jk}&\left(-\frac{\hbar}{4\pi T}\right) \\ &\times\{\ln|\sin[\tau'-\sigma'-(\tau-\sigma)]| + \ln|\sin[\tau'+\sigma'-(\tau+\sigma)]|\}.\end{aligned} \quad (5.6)$$

(Compare with [1, 2, 4, 34].)

5.2. Open strings

For open strings we get

$$\begin{aligned}\langle Q_m^j(\tau) * Q_n^k(\tau')\rangle &= \delta_{jk}\delta_{mn}\frac{\hbar}{2|m|}\exp(-\mathrm{i}|m|(\tau-\tau')) \\ \langle Q_m^j(\tau) * P_n^k(\tau')\rangle &= \delta_{jk}\delta_{mn}\frac{\mathrm{i}\hbar}{2}\exp(-\mathrm{i}|m|(\tau-\tau')) = -\langle P_m^j(\tau) * Q_n^k(\tau')\rangle \\ \langle P_m^j(\tau) * P_n^k(\tau')\rangle &= \delta_{jk}\delta_{mn}\frac{\hbar|m|}{2}\exp(-\mathrm{i}|m|(\tau-\tau')) \qquad m, n \neq 0.\end{aligned} \quad (5.7)$$

Then

$$\begin{aligned}\langle X^j(\tau,\sigma) * X^k(\tau',\sigma')\rangle = \delta_{jk}\Bigg\{&\langle x^j x^k\rangle + \frac{\mathrm{i}\hbar}{2\pi T}(\tau'-\tau) + \frac{\hbar}{\pi T} \\ &\times\sum_{n=1}^{\infty}\frac{\exp(-\mathrm{i}n(\tau-\tau'))\cos(n\sigma)\cos(n\sigma')}{n}\Bigg\}.\end{aligned} \quad (5.8)$$

Performing the summations in equation (5.8) and removing the part independent of the coordinates we get

$$\langle X^j(\tau,\sigma) * X^k(\tau',\sigma')\rangle \sim \delta_{jk}\left(-\frac{\hbar}{4\pi T}\right)\left\{\ln\left|\sin\left(\frac{\tau'-\sigma'-(\tau-\sigma)}{2}\right)\right| \right.$$
$$+\ln\left|\sin\left(\frac{\tau'-\sigma'-(\tau+\sigma)}{2}\right)\right|$$
$$\left. +\ln\left|\sin\left(\frac{\tau'+\sigma'-(\tau-\sigma)}{2}\right)\right| + \ln\left|\sin\left(\frac{\tau'+\sigma'-(\tau+\sigma)}{2}\right)\right|\right\}. \tag{5.9}$$

6. Final remarks

In this paper we have applied the Weyl–Wigner–Moyal formalism to bosonic string theory. As we have seen, this deformation quantization formalism provides us with a tool which enables one to describe the quantum bosonic string in terms of the deformed Poisson–Lie algebra. We have shown that the first quantized string system is equivalently seen as the Poisson bracket deformation of the bosonic classic string. We have found that, in order to make this deformation, the light-cone gauge allows the simplest description of the phase space $\mathcal{Z}$ (of the closed and open string) where this deformation is easily implemented.

With the identification of suitable phase space coordinates, the Moyal $*$ product is obtained (see equations (3.13), (3.19) and (3.41)). Then the SW quantizer was also computed and was used to find the Wigner functional equation (3.16). The corresponding Wigner functional of the ground state for closed and open strings was also computed (see equations (3.32) and (3.42), respectively). Oscillator variables (P, Q) introduced in section 2 greatly facilitate these computations. The normal ordering from the deformation quantization formalism was applied in section 3 to compute the Casimir energy in bosonic string theory. The Virasoro algebra with a central extension was also obtained within the deformed formalism (see equations (4.18) and (4.23)). Finally, as an application of the deformation quantization, the two-point correlation functions were also computed. Although up to here we have reproduced some known facts from string theory, we believe that deformation quantization possesses various advantages. One advantage of this scheme, with respect to the operator formalism, is that here the classical tools are still the relevant structure. Operators are no longer necessary and the Hilbert (Fock) space structure is encoded into the definition of the Moyal $*$ product. Thus the usual complications of operators are avoided and it facilitates the computation of relevant observables. After that, the formalism will be completely independent of the Hilbert (Fock) space structure and further generalization implies only the generalization of classical structures as the Moyal $*$ product. This is of great advantage because it enables us to use more general formalisms (such as the Fedosov one for curved phase space and the Kontsevich one for Poisson structures [19]) by generalizing only the $*$ product. In the classical theory, curved phase spaces would be artificial, but in field theory, the quantization of the theory requires the quantization of the moduli spaces of classical solutions of the field equation of motion. It is very well known that these spaces have non-trivial geometric structure (as Kähler or hyper-Kähler) and here the use of Fedosov's formalism seems to be more natural. In [18] and in this paper we intend to give the first steps toward a general programme of quantizing the moduli spaces of different classical field theories in various dimensions. Work in these directions will be reported elsewhere. Another advantage of the deformation quantization formalism is that the description of semiclassical effects is explicit and under control. Thus it might be useful in the description of semiclassical effects in string theory such as world-sheet instantons.

The extension of the formalism presented here to the interacting superstring theory and its further generalization to superstring field theory is our final goal. In our opinion, another interesting question is the description of the quantization of the non-commutative effective supersymmetric gauge on the brane, coming from string theory with a constant Neveu–Schwarz B field [11]. It is well known that the non-commutativity of the gauge theory is described through a star product which depends on this B field. Deformation quantization of this non-commutative gauge theory requires a further star product. That is, the *deformation of a deformation*. Deformation quantization is a promising formalism to describe both deformed structures within the same framework. The derivation of these structures from string theory is also very intriguing. Some work on defomation quantization of supersymmetric field theory (see e.g. [36–38]) seems to be crucial in searching for a solution to this problem. The most intriguing problem is to obtain a similar result for the case of string field theory for the bosonic and fermionic cases. It is known that the quantization of these systems (BRST cohomology, path integration, Batalin–Vilkovisky, etc) implying second class contraints can be addressed with the Dirac bracket. The Dirac bracket can be described through a Poisson structure and not by a symplectic one. Therefore it seems to us that correct degrees of freedom can be quantized by using the Kontsevich deformation quantization and its interplay with the Fedosov formalism. Finally, the relation of the formalism presented here to that of the covariant phase space approach worked out by Crnković and Witten [39] deserves further investigation.

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References

[1] Green M B, Schwarz J H and Witten E 1986 *Superstring Theory* 2 vols (Cambridge: Cambridge University Press)

[2] Polchinski J 1998 *String Theory* 2 vols (Cambridge: Cambridge University Press)

[3] Hatfield B 1992 *Quantum Field Theory of Point Particles and Strings* (New York: Addison-Wesley)

[4] Lüst D and Theisen S 1989 *Lectures on String Theory (Lecture Notes in Physics vol 346)* (Berlin: Springer)

[5] Yau S T (ed) 1987 *Mathematical Aspects of String Theory* (Singapore: World Scientific)

[6] Yau S T (ed) 1998 *Mirror Symmetry I* (Providence, RI: American Mathematical Society)
Greene B and Yau S T (ed) 1997 *Mirror Symmetry II* (Providence, RI: American Mathematical Society)

[7] Vafa C 1998 Geometric physics *Preprint* hep-th/9810149

[8] Witten E 1998 D-branes and K-theory *J. High Energy Phys.* 12(1998)019
Witten E 1998 *Preprint* hep-th/9810188

[9] Connes A, Douglas M and Schwarz A 1998 Noncommutative geometry and matrix theory: compactification on tori *J. High Energy Phys.* 02(1998)003
Connes A, Douglas M and Schwarz A 1997 *Preprint* hep-th/9711162

[10] Schomerus V 1999 D-branes and deformation quantization *J. High Energy Phys.* 06(1999)030

[11] Seiberg N and Witten E 1999 String theory and noncommutative geometry *J. High Energy Phys.* 09(1999)03
Seiberg N and Witten E 1999 *Preprint* hep-th/9908142

[12] Dito J 1990 *Lett. Math. Phys.* **20** 125
Dito J 1993 *Lett. Math. Phys.* **27** 73

[13] Antonsen F 1997 *Phys. Rev.* D **56** 920

[14] Antonsen F 1997 Deformation quantization of constrained systems *Preprint* gr-qc/9710021

[15] Antonsen F 1997 Deformation quantization of gravity *Preprint* gr-qc/9712012

[16] Curtright T, Fairlie D and Zachos C 1998 *Phys. Rev.* D **58** 025002

[17] Curtright T and Zachos C 1999 *J. Phys. A: Math. Gen.* **32** 771

[18] García-Compeán H, Plebański J F, Przanowski M and Turrubiates F J 1999 Deformation quantization of classical fields *Preprint* hep-th/9909206
[19] Fedosov B 1994 *J. Diff. Geom.* **40** 213
Kontsevich M 1997 Deformation quantization of Poisson manifolds I *Preprint* q-alg/9709040
[20] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1971 *Relativistic Quantum Theory* (New York: Pergamon)
[21] Weyl H 1931 *Group Theory and Quantum Mechanics* (New York: Dover)
[22] Wigner E P 1932 *Phys. Rev.* **40** 749
[23] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99
[24] Stratonovich R L 1956 *Sov. Phys.–JETP* **31** 1012
[25] Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2161
Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2206
[26] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 61
Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 111
[27] Tatarskii W I 1983 *Usp. Fiz. Nauk* **139** 587
[28] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[29] Cariñena J F, Gracia Bondía J M and Varilly J C 1990 *J. Phys. A: Math. Gen.* **23** 901
[30] Gadella M 1995 *Fortschr. Phys.* **43** 229
[31] Plebański J F, Przanowski M and Tosiek J 1996 *Acta Phys. Polon.* B **27** 1961
[32] Bordemann M, Neumaier N and Waldman S 1999 *J. Geom. Phys.* **29** 199
[33] Kay B S 1979 *Phys. Rev.* D **20** 3052
[34] Birrell N D and Davis P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[35] Isham C J 1978 *Proc. R. Soc.* A **362** 383
[36] Batalin I A and Fradkin E S 1988 *Ann. Inst. Henri Poincaré* **49** 145
[37] Ferrara S and Lledó M A 2000 Some aspects of deformation of supersymmetric field theories *Preprint* hep-th/0002084
[38] Terashima S 2000 A note on superfields and noncommutative geometry *Preprint* hep-th/0002119
[39] Witten E 1986 *Nucl. Phys.* B **276** 291
Crnković C and Witten E 1987 Covariant description of canonical formalism in geometrical theories *Three Hundred Years on Gravitation* ed S W Hawking and W Israel (Cambridge: Cambridge University Press) p 676
Crnković C 1987 *Nucl. Phys.* B **288** 419
Crnković C 1988 *Class. Quantum Grav.* **5** 1557